

National Centre of Competence in Research
Financial Valuation and Risk Management

Working Paper No. 859

The Finite-Time Horizon/Stochastic Interest Rate Jeanblanc-Shiryaev Model

Andrea Barth

Santiago Moreno-Bromberg

Oleg Reichmann

First version: July 2013
Current version: July 2013

This research has been carried out within the NCCR FINRISK project on
“Systemic Risk and Dynamic Contract Theory”

THE FINITE-TIME HORIZON/STOCHASTIC INTEREST RATE JEANBLANC-SHIRYAEV MODEL

ANDREA BARTH, SANTIAGO MORENO-BROMBERG, AND OLEG REICHMANN

ABSTRACT. In this paper the optimal consumption strategy of an investor who owns a fixed sized risky project is studied. The cash flows generated by the risky project follow an arithmetic Brownian motion, and the investor earns interest on cash reserves. The short-rate may be stochastic, and the time horizon may be finite. This results in a family of Hamilton-Jacobi-Bellman variational inequalities that include PDEs whose solutions must be approximated numerically. To do so an finite element approximation and a time marching scheme are employed.

1. Introduction

Starting with Merton's seminal contribution [13] in 1969, the study of problems of optimal portfolio choice in continuous time has produced a vast literature. One of the relatively younger branches of the such literature was initiated in 1995 with the works of Jeanblanc & Shiryaev [9] and Radner & Shepp [15], where the investor is assumed to be risk-neutral, the size of his risky project is fixed (i.e. there is no portfolio rebalancing) and he controls the level of his cash reserves by continuously choosing the amount he consumes. It is assumed that the cashflows produced by the project follow an arithmetic Brownian motion. All these ingredients combine to generate an non-standard stochastic control problem: The value function satisfies a Hamilton-Jacobi-Bellman variational inequality, where an ODE describes its behavior whenever no consumption takes place, and a complementarity condition describes the sets where consumption occurs. In this paper we take the model in [9] and extend it by assuming the investor has access to a savings account whose rate of return is stochastic (we use a CIR process to model the short-rate). Furthermore, we remove the stationarity assumption and allow for projects with finite lifespan. These two modifications result in HJB inequalities that include partial differential equations whose solutions must be approximated numerically. Moreover, the free-boundary

2000 *Mathematics Subject Classification.* 91G10; 91G30; 91G80.

Key words and phrases. Portfolio choice, singular stochastic control, numerical methods for PDEs.

The research leading to these results has received funding from the ERC (grant agreement 249415-RMAC) and from NCCR FinRisk (project Banking and Regulation).

problem of determining the consumption boundary is complicated by the multidimensionality of the problem in hand.

As mentioned above, the literature on optimal dividend distribution in a setting where investors are risk-neutral is quite vast. Højgaard & Taskar consider in [7] the problem of an insurance firm whose manager must also continuously choose the proportional amount of claims to be reinsured. Within a corporate-finance framework, Rochet & Villeneuve [8] solve the Merton-like problem of continuous, optimal structuring of a firm's balance sheet. The corresponding mathematical formulation is more complicated than that in [7], but surprisingly the structure of the value functions turns out to be quite similar. In [7] the state variable is the firm's level of cash reserves, whilst in [8] the firm's equity plays that role. Even though in both cases the control has two dimensions, the fact that there is only one state variable implies the problems can be (almost) fully analytically solved. The problem where the project has a fixed size but there is the possibility of refinancing the firm, should the cash reserves hit zero, was studied in the proportional-costs setting by Lokka & Zervos [12] and in the fixed-costs one by Descamp, Mariotti, Rochet & Villeneuve in [5]. Once again the models in these papers contemplate two control variables, yet only a single state one. More recently Jiang & Pistorius [10] and Akyildirim, Güney, Rochet & Soner in [2]) have studied models where the short-rate is assumed to evolve according to a Markov chain. To our knowledge these are the first two papers in this branch of literature where multidimensional state variables are considered, albeit one (the interest rate) has finitely many states. Our paper is an extension of sorts of [10] and [2]. We consider a short-rate that follows a Cox-Ingersoll-Ross (CIR) process, as well as finite-time horizons. We focus on the numerical approximation of the free boundaries where dividends are distributed, both in the no-savings and in the general setting. In the latter case we present a modification of the original Jeanblanc-Shiryaev model where the option to save expires later than the risky project. This generates an interesting problem in which the terminal condition for the value function is itself the value function of a second-step optimal control problem.

In terms of numerical methods, we use a finite element discretization in state space and a finite difference discretization in time of the arising PDEs. Using boundary conditions of the value function at the consumption boundary, we are able to approximate the value function numerically. We carry out a sanity check of our method comparing the numerical solution for projects with large life spans to the stationary limit, which is in certain cases available in closed form. The presented methodology has similarities to the pricing of American options, see e.g. [1] or [6].

The remainder of the paper is organized as follows: We describe in Section 2 the general, multidimensional problem we wish to tackle. In Section 3 we look at the finite-horizon setting without savings. The full model with a short-rate that follows a CIR process is presented in Section 4. A description of the numerical methods for PDEs that are used throughout the paper can be found in Section 5, after which we conclude.

2. General Model

We consider the problem of an individual who manages a fixed-size, risky project over a finite time horizon $[0, T]$. In order to describe the evolution of the investor's cash reserves, let us introduce the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. The process $W(t) = (W_1(t), W_2(t))$ is a standard, two-dimensional \mathbb{P} -Brownian motion that generates the filtration $\{\mathcal{F}_t\}$. Let μ , c_0 and σ be greater than zero. The project's cashflows follow the diffusion process

$$dC(t) = \mu dt + \sigma dW_1(t), \quad C(0) = c_0.$$

The investor is risk-neutral, and his objective is to maximize the project's value, defined at each date as the discounted, expected consumption stream over the remaining life of the project¹. To this end, he must decide on his consumption at each date $t \in [0, T]$. A CUMULATIVE CONSUMPTION PROCESS is any non-decreasing, adapted and càglàd process $\{L(t)\}_{t \geq 0}$. The admissible set of all such processes will be denoted by \mathcal{A} . For a given $L \in \mathcal{A}$, the dynamics of the corresponding CASH RESERVES PROCESS are described by the SDE

$$dR^L(t) = (\mu + r(t)R^L(t))dt + \sigma dW(t) - dL(t), \quad R^L(0) = x_0.$$

Here $r(t)$ denotes the stochastic short-rate, which evolves according to

$$dr(t) = m(t, r(t))dt + s(t, r(t))dW_2(t), \quad r(0) = r_0.$$

In other words, the investor's cash reserves increase both via the project's cashflows and the interest accrued, at rate r , on non-consumed wealth². On the other hand, the cash reserves decrease whenever consumption takes place. The bankruptcy time τ associated to a consumption strategy L is defined as

$$\tau_L := \inf\{T \geq t > 0 \mid R^L \leq 0\}.$$

Let us denote by ρ the investor's discount rate. For a given choice of L , the investor's expected, discounted consumption at date t is given by the quantity

$$V^L(t, x, r) := \mathbb{E}_t \left[\int_t^{\tau_L} e^{\rho(t-s)} dL(s) \right],$$

¹Within a corporate-finance setting, one would talk about the discounted, expected DIVIDENDS stream.

²In most of the related models in the literature r is deterministic; two notable exceptions being [10] and [2].

the operator $\mathbb{E}_t[\cdot]$ is the expectation conditional on $R^L(t) = x$ and $r(t) = r$. We analyze the problem through the properties of the value function

$$V(t, x, r) := \sup_{L \in \mathcal{A}} V^L(t, x, r). \quad (1)$$

Here t , x and r are the state variables and L is the control. Unlike the standard stochastic control techniques, the fact that the investor is risk-neutral implies that L needs not be absolutely continuous. This sets the problem within the realm of Singular Stochastic Control (see, e.g. [14]) and leads to a Hamilton–Jacobi–Bellman (HJB) variational inequality. The latter includes a PDE in the three state variables, whose solution must be approximated numerically.

3. The Particular Case $r \equiv 0$

In order to fix ideas, we first center our attention in the particular case where the investor earns no interest on accumulated reserves. This is exactly the finite-horizon version of the problem studied in [9], where the authors show the optimal strategy is as follows: refrain from consumption as long as the cash reserves remain under a certain threshold x^* ; whenever the current reserves level x is greater than x^* , consume $x - x^*$ immediately. Due to the fact that cash flows are modeled as a continuous process, only at date $t = 0$ may the level of reserves be larger than x^* , when an exceptional lump-sum of size $x_0 - x^*$ would be consumed. Given that we are moving away from stationarity, instead of a consumption level x^* , we will have a consumption boundary $x^*(t)$, for $t \in [0, T]$. For a given consumption strategy L , the investor's assessment of the project's value is

$$V^L(t, x) = \mathbb{E}_t \left[\int_t^{\tau_L} e^{\rho(t-s)} dL(s) \right],$$

and the corresponding value function is then $V(t, x) := \sup_{L \in \mathcal{A}} V^L(t, x)$. Within this section we work under the following

Assumption 3.1. *The project has zero liquidation value, and any cash reserves remaining in the project at date T are immediately consumed by the investor. In other words, $dL(T) = R^L(T)$.*

We have the following

Theorem 3.2. *Assume the mapping $(t, x) \mapsto V(t, x)$ is jointly concave and twice continuously differentiable in the space variable x . Then the following HJB variational inequality holds in the strong sense:*

$$\max \left\{ \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} + \mu \frac{\partial V}{\partial x} - \rho V + \frac{\partial V}{\partial t}, 1 - \frac{\partial V}{\partial x} \right\} = 0, \quad (2)$$

together with the boundary condition $V(t, 0) = 0$ for all $t \in [0, T]$, and the terminal condition $V(T, x) = x$.

A proof of Theorem 3.2 can be found in [14]. The intuition behind it is the following: The left hand term in Inequality 2 is a standard HJB equation, and it represents the first order conditions arising from the Itô formula. From the dynamic programming principle, the right hand term generates an expression of the form

$$\mathbb{E}_t \left[\int_t^{\tau_L} e^{\rho(t-s)} \left(1 - \frac{\partial V}{\partial x}(R^L(s)) \right) dL(s) \right]$$

in the integral representation of $V(x, t)$. It can be shown that the value function's first partial derivative in the space direction is greater or equal than one, thus the expression above is non-positive (since L is a non-decreasing process). This suggests that the mass of the measure $dL(s)$ is concentrated on the set $\{\frac{\partial V}{\partial x} R^L(s) = 1\}$, thus the localized nature of the optimal control. The boundary and terminal conditions are a direct consequence of Assumption 3.1. The consumption barrier $x^*(t)$ satisfies

$$1 = \frac{\partial V}{\partial x}(x^*(t), t) \quad \text{and} \quad \frac{\partial^2 V}{\partial x^2}(x^*(t), t) = 0.$$

The (complementary) condition on the first partial derivative in the space direction, i.e. in the x -variable, indicates that at $x^*(t)$ the marginal value of one unit of cash in the project equals the marginal value of a unit of cash that is immediately consumed. The supercontact condition in the space dimension is standard in the stochastic control literature, where it is deemed a necessary condition for optimality (see, for example [7] and [9]), and it is essential for numerical approximations.

Remark 3.3. *Inequality (2) imposes no restriction on the regularity properties of V in the r -direction. Problem (10) introduced below has certain structural similarities with a pricing problem of an American option under stochastic volatility, it corresponds to Problem (2) with a stochastic short rate following a CIR process. The pricing of American options as well as the solution of Problem (10) involves finding a free boundary and solving different differential equations on both sets. In both cases we have a second background process, the stochastic volatility or the stochastic short rate. Despite these similarities, the behavior of the free boundary with respect to the initial condition of the background process differs significantly. While we observe a smooth dependence in the case of option pricing, Problem (10) might lead to discontinuous free boundaries as observed in Figure 2.*

3.1. The stationary case. We briefly present in this section, for the sake of completeness and to have a point of reference, the solution to the stationary version (that in [9]) of the problem. In other words, the infinite-horizon case, where the only state variable is the current reserves

level x . In such setting the HJB variational inequality satisfied by $V(\cdot)$ is

$$\max \left\{ \frac{1}{2}V'' + \mu V' - \rho V, 1 - V' \right\} = 0.$$

Let x^* be the consumption boundary. On $(0, x^*)$ any candidate value function must satisfy $V' > 1$ and it is found by solving the linear, second order ODE $\frac{1}{2}V'' + \mu V' - \rho V = 0$, whose general solution is

$$V(x) = b_1 e^{r_1 x} + b_2 e^{r_2 x}, \quad \text{where } r_i := \frac{-\mu + (-1)^i \sqrt{\mu^2 + 2\sigma^2 \rho}}{\sigma^2}. \quad (3)$$

For each $x^* > 0$ we define V_{x^*} as the particular solution to Equation (3) that satisfies $V'_{x^*}(x^*) = 1$ and $V''_{x^*}(x^*) = 0$. It is not complicated to show that

$$V_{x^*}(x) = \frac{1}{r_1 r_2} \left(\frac{r_2^2}{r_2 - r_1} e^{r_1(x-x^*)} - \frac{r_1^2}{r_2 - r_1} e^{r_2(x-x^*)} \right). \quad (4)$$

Assumption 3.1 implies that $V_{x^*}(0) = 0$, hence x^* can be found by solving

$$\frac{r_2^2}{r_2 - r_1} e^{-r_1 x^*} = \frac{r_1^2}{r_2 - r_1} e^{-r_2 x^*} \Rightarrow x^* = \frac{2}{r_1 - r_2} \log \left(-\frac{r_2}{r_1} \right) > 0. \quad (5)$$

On $[x^*, \infty)$ the value function is linear:

$$V(x) = (x - x^*) + \frac{\mu}{\rho}.$$

Notice this means that for $x > x^*$, the project's value is the immediate consumption $x - x^*$ plus the discounted value of a bond paying a continuous dividend μdt per unit of time. Once x^* has been found, it can be shown that the optimal $\{L^*(t)\}_{t \geq 0}$ is determined by solving the following Skorohod problem on $[0, x^*]$: Let the processes (R^*, L^*) be a solution to

$$R^*(t) = x + \int_0^t \mu ds + \int_0^t \sigma dW(s) - L^*(t), \quad (6)$$

$$0 \leq R^*(t) \leq x^*, \quad t \geq 0, \quad (7)$$

$$\int_0^\infty \mathbb{1}_{\{R^*(t) < x^*\}} dL^*(t) = 0, \quad (8)$$

where $\mathbb{1}_{\{\cdot\}}$ is the zero-one indicator function. A comprehensive treatise on such reflection problems can be found in [3] and [11]. The process $\{L^*(t)\}_{t \geq 0}$ is the local time of $\{R^*(t)\}_{t \geq 0}$ at level x^* (See [4] for a thorough exposition of Brownian local times). The impact of $\{L^*(t)\}_{t \geq 0}$ on the dynamics of R^* is to reflect the latter so that it remains under x^* . From Equation (8) we see that the mass of the measure $dL^*(t)$ is carried by the sets $\{R^*(t) = x^*\}$ thus $L^*(t)$ is inactive whenever $R^*(t) < x^*$.

3.2. The time-dependent case. To our knowledge no closed form solutions are available for the time-dependent case, therefore numerical methods have to be employed in order to approximate the solution of Problem (2). We employ a finite element discretization in the spatial domain and a finite difference scheme in time. We use the Crank–Nickolson or θ -scheme. We solve Equation (9)

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \mu\frac{\partial V}{\partial x} - \rho V + \frac{\partial V}{\partial t} = 0, \quad (9)$$

for each time-step on a sufficiently large computational domain with appropriate boundary conditions and in each time-step adjust the solution accordingly to the complementarity condition in Inequality (2). Details on the numerical algorithm are discussed in Section 5.

For our numerical experiment we consider the following parameters: $\mu = 0.3$, $\sigma = 0.3$ and $\rho = 0.15$. The corresponding free boundary for the stationary case is 0.7172, which can be obtained from Equation (5). These values were chosen for exposition purposes, the behavior of the solutions does not change qualitatively with different parameter choices. The optimal strategy is also localized, which is a consequence of the term $1 - \partial V/\partial x$ in Inequality (2). The optimal cumulative-consumption process $L^*(\cdot, t)$ is the local time of $R^*(t)$ at level $x^*(t)$. We plot the consumption boundary x^* as a function of time-to-maturity in Figure 1.

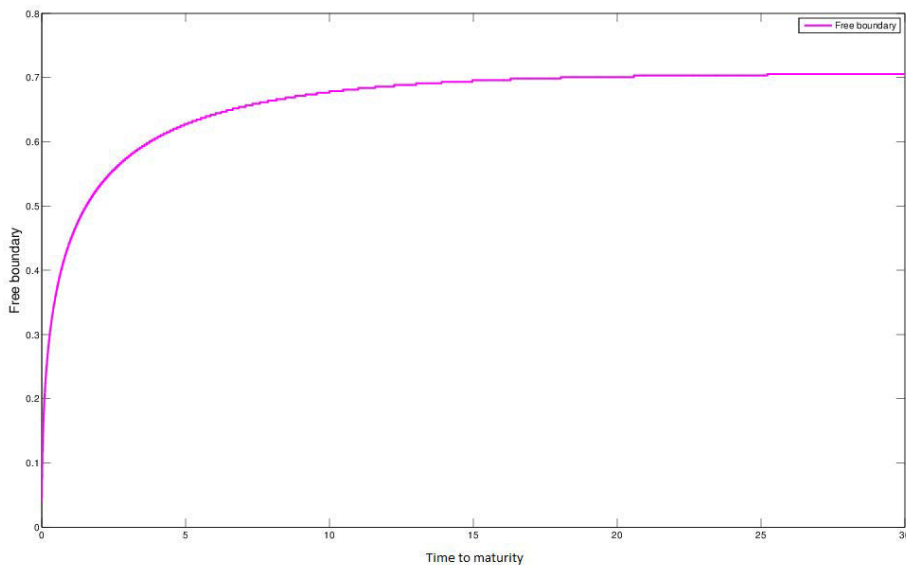


FIGURE 1. Free boundary for the $r = 0$ model

Notice that for large times $T \gg 1$ the free boundary approaches the stationary limit. On the other hand, for small times-to-maturity $x^*(t)$ decreases rapidly towards zero. This is consistent

with our assumption of immediate consumption of $R^L(T)$ at maturity: Due to discounting and given the approaching maturity, consumption becomes more enticing at lower reserves levels. We study a different final condition in Section 4.2.2, which naturally results in a different structure of the free boundary close to maturity.

4. Stochastic (CIR) Short-rate Model

In this section we consider both the stationary and the general settings under the assumption that the short rate evolves according to a Cox–Ingersoll–Ross (CIR) process with dynamics

$$dr(t) = a(b - r(t))dt + \xi\sqrt{r(t)}dW_2, \quad r(0) = r_0 > 0,$$

where a, b and ξ are positive. A mean-reverting process is a natural choice to model the short rate. The level of mean-reversion b , as well as its speed a have a stark influence on the structure of the consumption boundary. An important departure from the $r \equiv 0$ case presented in Section 3 is that whenever $r(t) > \rho$ no consumption will take place, regardless of the time to maturity or the cash reserves level. This occurs because the investor’s impatience is outweighed by the high interest paid on savings. Notice that this implies that for any $t < T$, the consumption barrier $x^*(r(t), t) = \infty$ whenever $r(t) > \rho$.

4.1. The stationary case. We briefly discuss the case $T = \infty$, where the value function satisfies the following HJB variational inequality:

$$\max \left\{ -\rho V + a(b - r) \frac{\partial V}{\partial r} + (\mu + rx) \frac{\partial V}{\partial x} + \frac{1}{2} \xi^2 r \frac{\partial^2 V}{\partial r^2} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2}, 1 - \frac{\partial V}{\partial x} \right\} = 0,$$

together with the boundary condition $V(0, r) = 0$. For values of $r < \rho$ the consumption barrier $x^*(r)$ satisfies

$$1 = \frac{\partial V}{\partial x}(x^*(r), r), \quad \frac{\partial^2 V}{\partial x^2}(x^*(r), r) = 0.$$

As discussed above, we expect to have

$$\lim_{r \nearrow \rho} x^*(r) = \infty.$$

An important observation is that $r(t) < \rho$ does not imply immediate consumption. This follows two complementary causes: First, for relatively low levels of cash reserves $R^L(t)$, consumption might increase the risk of bankruptcy to unacceptable levels; second, if the level of mean reversion is above ρ the future expectation of the interest rate level out weights the investor’s current impatient. We introduce a finite time horizon in Section 4.2, in which case a high speed of mean reversion plays the role of stationarity in terms of reducing the investor’s urge to consume.

4.2. The finite time–horizon case. For this section we have chosen two different approaches regarding the terminal condition. The first one is simply the “end-of-the-World” assumption made in Section 3, i.e. contingent on $\tau_L \geq T$, the amount $R^L(T)$ is consumed at maturity and $V(T, R^L(T), r(T)) = R^L(T)$. This is a natural assumption if the investor is actually the manager of a firm that had an a priori determined lifespan. In the case of a private, albeit sophisticated, investor, however, one could ask themselves why is it that the “saving” opportunity expires simultaneously with the risky project. Therefore, as a second case we analyze the situation where the savings account is accessible up to some time T_s that is strictly greater than T . The former case is mathematically simpler to tackle, but it exhibits a discontinuity of the consumption boundary at maturity. The latter avoids this issue, but it requires us to solve an additional control problem after date T to determine $V(T, R^L(T), r(T))$.

4.2.1. The single-regime setting. In this case the description of the value function $V(t, x, r)$ given in Section 2 is applicable as is. The HJB variational inequality is

$$\max \left\{ -\rho V + a(b-r) \frac{\partial V}{\partial r} + (\mu + rx) \frac{\partial V}{\partial x} + \frac{1}{2} \xi^2 r \frac{\partial^2 V}{\partial r^2} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial t}, 1 - \frac{\partial V}{\partial x} \right\} = 0, \quad (10)$$

with the boundary condition $V(t, 0, r) = 0$ for all $t \leq T$ and all $r \geq 0$. Whenever $r < \rho$ the consumption boundary satisfies

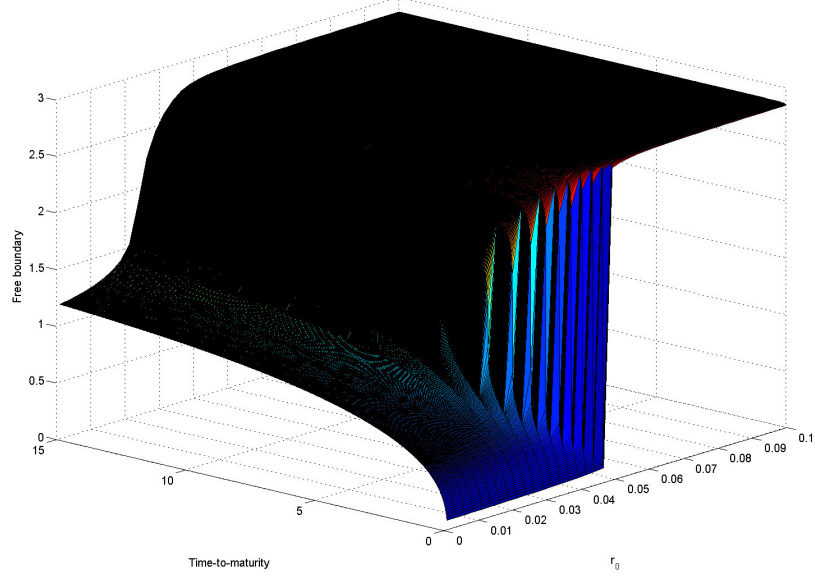
$$1 = \frac{\partial V}{\partial x}(x^*(r, t), r, t) \quad \text{and} \quad \frac{\partial^2 V}{\partial x^2}(x^*(r, t), r, t) = 0.$$

As discussed above, we expect

$$\lim_{r \nearrow \rho} x^*(r, t) = \infty \quad \forall t < T,$$

and $x^*(r, t) = \infty$ for all $r > \rho$. The source of the discontinuity of the consumption barrier at $t = T$ is the imposition of mandatory consumption at say date: If at $t = T - dt$ we have $r(T - dt) \gg \rho$ the investor postpones consumption in order to benefit from the favorable conditions of the savings account, thus $x^*(r(T - dt), T - dt) = \infty$. However, $R^L(T)$ must be consumed, which implies $V(T, x, r) = x$ and $x^*(r, T) = 0$ for all $r \geq 0$. In Figure 2 we plot x^* for $\rho = 0.05$, $\sigma = 0.3$, $a = 0.1$, $b = 0.05$; $\xi = 0.2$ and $\mu = 0.1$, where the high, flat section of the graph represents $x^* = \infty$. The corresponding value function is presented in Figure 3.

4.2.2. The two-regimes setting. To model the situation where the investor may still make use of the savings account after the risky project has expired, we introduce an additional maturity date $0 < T < T_s \leq \infty$. Over the period $[0, T]$ the investor faces a problem similar to that studied in Section 4.2.1, but the terminal conditions of the HJB variational inequality are different. Namely, on $[T, T_s]$ the investor may invest in and consume from the savings account,

FIGURE 2. Free boundary for the mandatory-consumption-at- $t = T$ model

and the value at time $t = T$ of this opportunity will be used as the terminal (in some sense intermediate) condition $V(T, x, r)$. We follow a dynamic programming approach and first solve the investor's problem on $[T, T_s]$. To this end let $M^K(t)$ be the cash-reserves level for $t \geq T$. Given a cumulative-consumption strategy K , then

$$dM^K(t) = r(t)M^K(t)dt - dK(t), \quad M^K(0) = R^L(T).$$

The value of $R^L(T)$ at maturity is then

$$U(R^L(T), r(T), T) := \sup_{K \in \mathcal{A}} \mathbb{E}_T \left[\int_0^\eta e^{-\rho s} dK(s) \right],$$

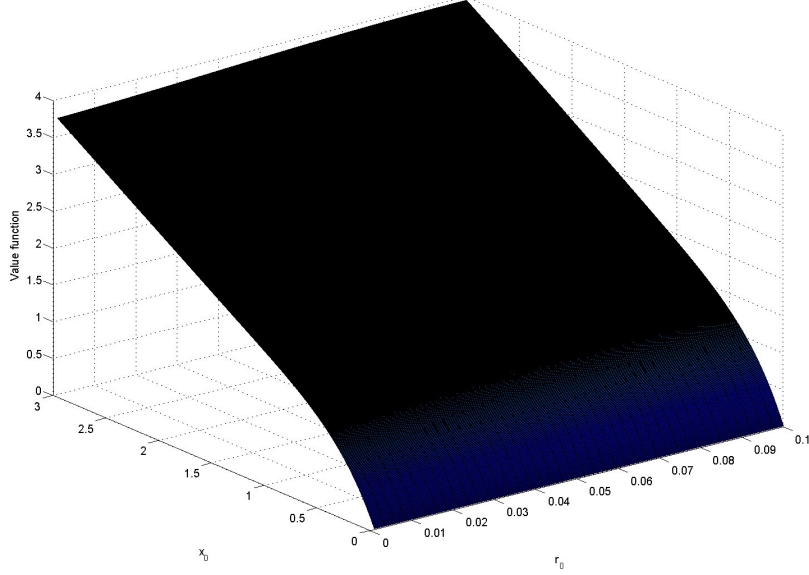
where

$$\eta := \inf \{T_s \geq t > T | M^K(t) \leq 0\}.$$

Below we consider $T_s < \infty$ and use the representation results in Section 4.2.1 with $\mu = \sigma = 0$ to obtain a variational inequality for U ³:

$$\max \left\{ -\rho U + a(b-r) \frac{\partial U}{\partial r} + rx \frac{\partial U}{\partial x} + \frac{1}{2} \xi^2 r \frac{\partial^2 U}{\partial r^2}, 1 - \frac{\partial V}{\partial x} \right\} = 0,$$

³One could of course also look at the stationary version of the problem and use the methodology of Section 4.1.

FIGURE 3. Value function for the mandatory-consumption-at- $t = T$ model

with terminal condition $U(m, r, T_s) = m$ (immediate consumption at maturity) and $U(0, r, t) = 0$ for all $t \in (T, T_s]$ and $r \geq 0$. Very interestingly, the optimal consumption strategy k^* turns out to be a bang-bang one: Consumption is postponed for large levels of r ; for a given $t \in (T, T_s]$, full consumption occurs whenever the short-rate falls below a certain threshold that depends on the speed a and the level b of mean reversion. The larger a is, the less relevant $r(t)$ is, and the more importance b has. For large times-to-maturity and $b > \rho$ there is no consumption. This behavior extends to mid-length maturities if a is large. On the other hand, for dates close to maturity and slow mean-reversion immediate consumption takes place almost immediately after the level of $r(t)$ becomes smaller than ρ .

Given a consumption strategy L and contingent on $R^L(t) = x$ and $r(t) = r$ we have that the project's value at date $t < \tau_L$ is

$$V^L(t, x, r) = \mathbb{E}_t \left[\int_t^{\tau_L} e^{\rho(t-s)} dL(s) + e^{-\rho(T-t)} U(R^L(T)) \right].$$

For $t < T$ the value function $V(t, x, r) := \sup_{L \in \mathcal{A}} V^L(t, x, r)$ satisfies the Variational Inequality (10) with boundary condition $V(t, 0, r) = 0$, and terminal condition becomes $V(T, x, r) =$

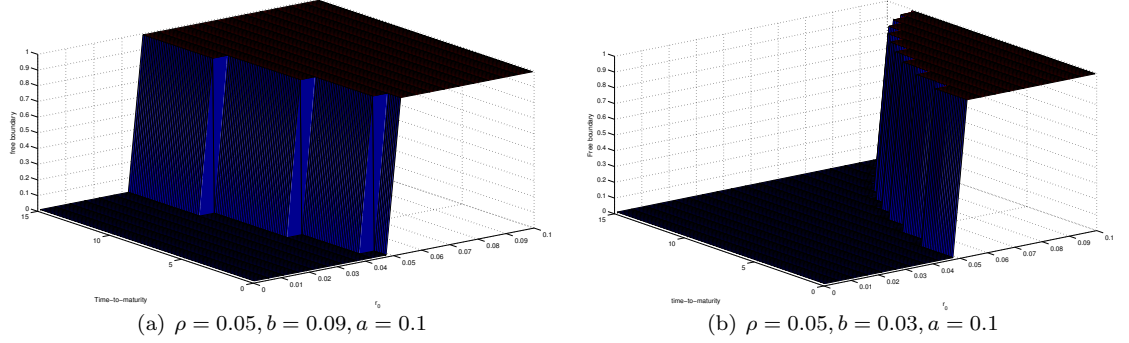


FIGURE 4. Free boundaries for a mean reversion speed of $a = 0.1$ and different levels of mean reversion

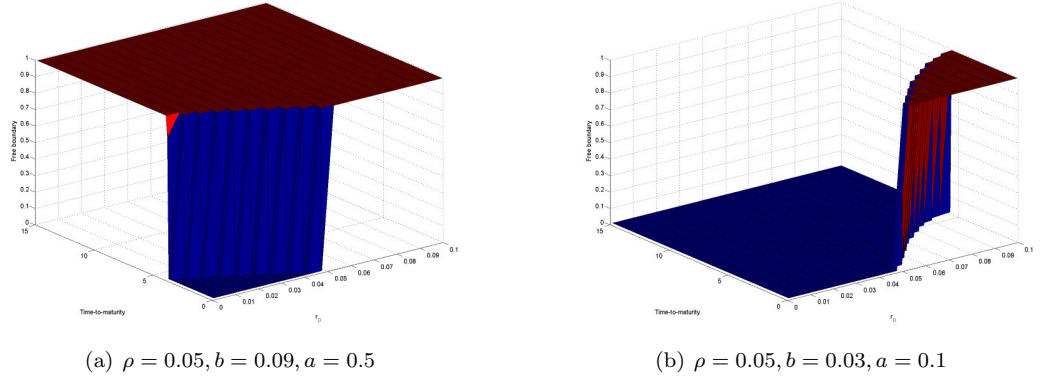
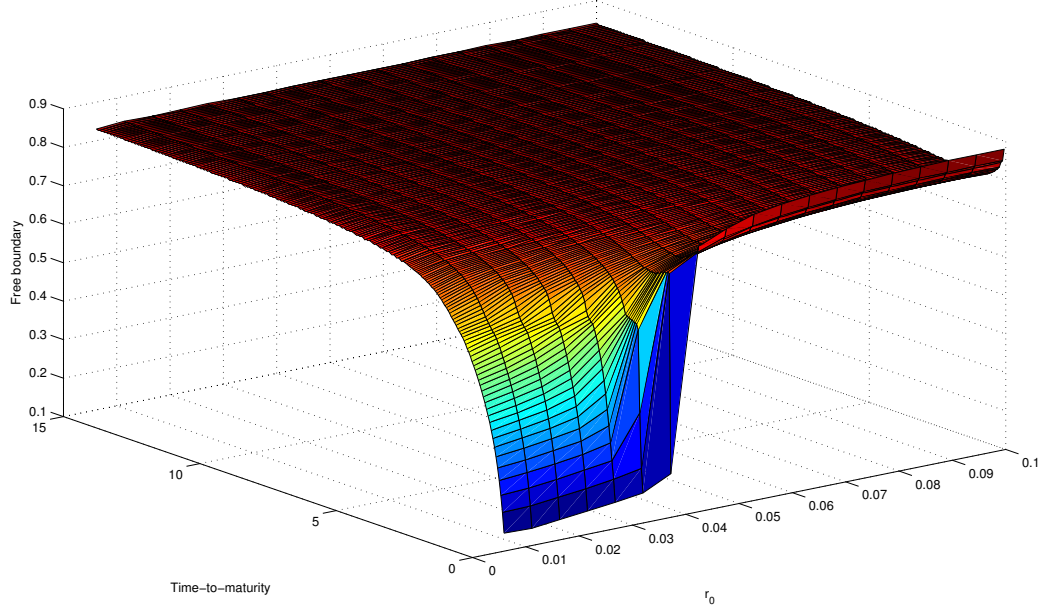


FIGURE 5. Free boundaries for a mean reversion speed of $a = 0.5$ and different levels of mean reversion

$U(x, r)$. In Figure 4.2.2 we plot the consumption boundary over $[0, T]$, i.e. time-to-maturity equal to zero corresponds to $t = T$. The jump at such time corresponds to that in Figure 4(a).

5. The Numerical Methods

In this section we discuss the numerical approximation of the solutions of Equations (2) and (10). Let us first consider the univariate setup of Equation (2). The first step in the approximation procedure is a time-discretization. Here we discuss the implicit Euler scheme for simplicity but any other time-stepping method, such as the θ -scheme or the hp -dG-timestepping scheme would also be applicable. Let us denote by $\mathcal{T} = \{t_0, \dots, t_m\}$ a partition of the time-interval $[0, T]$ with $t_0 = 0$ and $t_m = T$, $T > 0$ and $m \geq 1$. Approximating the

FIGURE 6. The consumption boundary up to the intermediate time T

time-derivative by the corresponding difference quotient we obtain the following sequence of problems:

$$V(t_m, x) = x$$

$$\max\left\{\frac{1}{2}\sigma^2\frac{\partial^2 V(t_i, x)}{\partial^2 x} + \mu\frac{\partial V}{\partial x} - \rho V(t_i, x) + \frac{V(t_{i+1}, x) - V(t_i, x)}{t_{i+1} - t_i}, 1 - \frac{\partial V(t_i, x)}{\partial x}\right\} = 0,$$

for $i = m-1, \dots, 0$. As a further approximation step we consider in each time-step i the problem

$$\frac{1}{2}\sigma^2\frac{\partial^2 \hat{V}(t_i, x)}{\partial^2 x} + \mu\frac{\partial \hat{V}(t_i, x)}{\partial x} - \rho\hat{V}(t_i, x) + \frac{V(t_{i+1}, x) - \hat{V}(t_i, x)}{t_{i+1} - t_i} = 0 \quad (11)$$

The function $\hat{V}(t_i, x)$ can be approximated by a finite-difference or finite element-discretization of Equation (11). This is standard and therefore not discussed here. We refer the interested reader to [1] or [6].

Having obtained an approximation of $\hat{V}(t_i, x)$ which we again by a slight abuse of notation denote by $\hat{V}(t_i, x)$ and $\hat{x}_i = \max_{x \in A_i} x$, with $A_i = \{x | \partial_x \hat{V}(t_i, x) \geq 1\}$ we obtain an approximation of $V(t_i, x)$ as

$$V(t_i, x) = \begin{cases} \hat{V}(t_i, x) & \text{if } \partial_x \hat{V}(t_i, x) \geq 1 \\ \hat{V}(t_i, \hat{x}_i) + (x - \hat{x}_i) & \text{if } \partial_x \hat{V}(t_i, x) < 1. \end{cases} \quad (12)$$

The methodology employed in the discretization of Problem (11) is analogous to the methods used for the approximation of the solution of pricing equations in the context of option pricing. The case of a stochastic interest is similar, with respect to the discretization methodology to a pricing equation with stochastic volatility.

The discretization of Equation (10) differs only slightly from the described procedure, as a two dimensional problem has to be solved in each time-step and the corresponding free boundary has a more involved structure.

6. Conclusions

We have extended the model of optimal dividends distribution in [9] and [15] to allow for a finite time horizon and/or interest being accrued on cash reserves at a stochastic rate. Given that say extension results in singular stochastic control problems with two or three state variables, we have made use of numerical methods for PDEs to approximate the value functions and, more interestingly, the free boundaries corresponding to the dividend boundaries. In the setting with savings, using a CIR process to model the evolution of the short-rate, we have seen that the investor's behavior heavily influenced by the speed and level of mean-reversion. This is particularly evident in our two-regimes setting when determining the terminal condition for the value function corresponding to the first period. For further research we have left the extension of the corporate-finance model of [8] where there is a varying level of debt to be serviced.

References

1. Y. Achdou and O. Pironneau, *Computational methods for option pricing*, Frontiers in Applied Mathematics, Society for Industrial and Applied Mathematics, 2005.
2. Güney I. E. Rochet J-C. Akyildirim, E. and H. M. Soner, *Optimal dividend policy with random interest rates*, Swiss Finance Institute Research Paper No. 13-14 (2013) (English).
3. R. F. Bass and P. Hsu, *The semimartingale structure of reflecting Brownian motion*, Proc. Amer. Math. Soc. **108** (1990), no. 4, 1007–1010. MR 1007487 (91a:60205)
4. Revouz D. and Yor M., *Continuous martingales and brownian motion.*, Springer, Germany, 1999.
5. Mariotti T. Rochet J-C. Descamps, J-P. and S. Villeneuve, *Free cash flow, issuance costs, and stock prices*, The Journal of Finance (2011), no. 66, 15011544.
6. N. Hilber, O. Reichmann, Ch. Schwab, and Ch. Winter, *Computational methods for quantitative finance: Finite element methods for derivative pricing*, Springer Finance, Springer London, Limited, 2013.
7. B. Hoejgaard and M. Taksar, *Controlling risk exposure and dividends pay-out schemes: Insurance company example*, Mathematical Finance **9** (1998), 153–182.
8. Rochet J-C. and Villeneuve S., *Corporate portfolio management*, Annals of Finance **1** (2005), no. 3, 225–243.
9. M. Jeanblanc-Picqué and A. N. Shiryaev, *Optimization of the flow of dividends*, Uspekhi Mat. Nauk **50** (1995), no. 2(203), 25–46.

10. Z. Jiang and M. Pistorius, *Optimal dividend distribution under markov regime switching*, Finance and Stochastics **16** (2012), no. 3, 449–476 (English).
11. Lehoczy J. Ramanan K. Kruk, L. and S. E. Shreve, *An explicit formula for the skorokhod map on $[0, a]$* , Ann. Probab. **35** (2007), no. 5, 1740–1768. MR 2349573 (2008k:60087)
12. A. Lokka and M. Zervos, *Optimal dividend and issuance of equity policies in the presence of proportional costs*, Insurance mathematics and economics **42** (2008), no. 3, 954–961.
13. R. C. Merton, *Lifetime portfolio selection under uncertainty: The continuous-time case*, The Review of Economics and Statistics **51** (1969), no. 3, 247–57.
14. H. Pham, *Continuous-time stochastic control and optimization with financial applications*, Stochastic Modelling and Applied Probability, Springer, 2010.
15. R. Radner and L. Shepp, *Risk vs. profit potential: A model for corporate strategy*, Journal of Economic Dynamics and Control (1996), 1371–1393 (English).